Lecture 3 - Objective and Outcomes

Moments provide summary information about distributions. Appropriate conditioning allows simple solutions to apparently intractable problem.

- moments and moment generating functions
- cumulant generating functions
- functions of random variables

After reviewing the notes you should:

- be able to work of mgf and cgf for a variety of different distributions
- understand a simple method for generating draws from random variables
- be able to get marginal distributions from joint distributions

Properties of moments

1. $m_1 = E(X) = \mu$ (mean); $\mu_1 = 0$.

2. $\mu_2 = E(X^2) - E(X)^2 = \mathrm{var}(X) = \sigma^2$ (variance).

3. Coefficient of skewness: $\gamma_1 = E[(X-\mu)^3]/\sigma^3 = \mu_3/\mu_2^{\frac{3}{2}}$.

4. Coefficient of kurtosis: $\gamma_2 = (E[(X-\mu)^4]/\sigma^4) - 3 = (\mu_4/\mu_2^2) - 3$.

Bivariate distribution function

1.
$$F(-\infty, y) = \lim_{x \to -\infty} F(x, y) = 0,$$
$$F(x, -\infty) = \lim_{y \to -\infty} F(x, y) = 0,$$
$$F(\infty, \infty) = \lim_{x \to \infty, y \to \infty} F(x, y) = 1.$$

2. $P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$
$= F(x_2, y_2) - F(x_1, y_2) - [F(x_2, y_1) - F(x_1, y_1)]$.

3. Right continuous is x:
$\lim_{h\downarrow 0} F(x+h,y) = F(x,y)$,
Right continuous in y:
$\lim_{h\downarrow 0} F(x,y+h) = F(x,y)$.

Marginal distribution functions

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y) = F_{X,Y}(x, \infty),$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y) = F_{X,Y}(\infty, y).$$

Discrete bivariate distributions

1. Joint mass function:
   $f(x, y) = f_{X,Y}(x, y) = P(X = x, Y = y)$.

2. Probabilities:
   $P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$
   $= \sum_{x_1 < x \leq x_2} \sum_{y_1 < y \leq y_2} f(x, y)$.

3. Marginal mass functions:
   $f_X(x) = \sum_y f_{X,Y}(x, y)$,
   $f_Y(y) = \sum_x f_{X,Y}(x, y)$.

Continuous bivariate distributions

1. Joint density function: is a positive real-valued function f such that

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)dudv \quad \text{for each } x,y \in \mathbb{R}.$$

2. Probabilities:
   $P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy.$

3. Marginal density functions:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx,$$

Expectation of a function of two variables

$$E[g(X,Y)] = \begin{cases} \sum_y \sum_x g(x,y) f_{X,Y}(x,y), & \text{discrete case,} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy, & \text{continuous case.} \end{cases}$$

Discrete conditional distributions

:

1. Conditional distribution of Y given $X = x$: $F_{Y|X}(y|x) = P(Y \leq y|X = x)$.

2. Conditional mass of Y given $X = x$: $f_{Y|X}(y|x) = P(Y = y|X = x) = f_{X,Y}(x, y)/f_X(x)$.

3. Relationship between distribution and mass:
   $F_{Y|X}(y|x) = \sum_{y_i \le y} f_{Y|X}(y_i|x)$.

Continuous conditional distributions

1. Conditional distribution of Y given $X = x$:
   $F_{Y|X}(y|x) = \int_{-\infty}^{y} (f_{X,Y}(x, v)/f_X(x))dv$.

2. Conditional density of Y given $X = x$:
   $f_{Y|X}(y|x) = f_{X,Y}(x, y)/f_X(x)$.

Conditional, joint and marginal densities

1. Conditional mass/density
   $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
   $$= \begin{cases} \frac{f_{X,Y}(x,y)}{\sum_y f_{X,Y}(x,y)}, & \text{discrete case,} \\ \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy}, & \text{continuous case.} \end{cases}$$

2. Joint mass/density:
   $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$.
3. Marginal mass/density

$$f_Y(y) = \begin{cases} \sum_x f_{Y|X}(y|x) f_X(x), & \text{discrete case,} \\ \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx, & \text{continuous case.} \end{cases}$$